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NOTE

GRAPHICAL EULERIAN NUMBERS AND CHROMATIC GENERATING FUNCTIONS

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In this paper some further properties of the coefficients of a chromatic generating function introduced by Linial are proved. A combinatorial interpretation of these numbers is given by specializing some results of Stanley on posets to surjective n-colorings of a graph G of order n compatible with linear orders on $V(G)$ which extend acyclic orientations of G.

If G is composed from n isolated vertices these coefficients are classical Eulerian numbers $A(n, k)$.

Let G be a finite graph having vertex set $V(G)=\{x_1, \ldots, x_n\}$, edge set $E(G)$ and chromatic number $\chi(G)=k$. A p-coloring of G is a partition of $V(G)$ into p classes such that any two vertices in the same class are not adjacent. The number of p-colorings of G is denoted $\mathrm{Col}_p(G)$. By $P_G(x)$, or simply $P(x)$, we denote the chromatic polynomial of G. In [1] Linial introduced the chromatic generating function of G by

$$F_G(x)=\sum_{n=0}^{\infty} P(n)x^n$$

and proved that $F_G(x)=Q(x)/(1-x)^{n+1}$, where $Q(x)$ is a polynomial of degree n having nonnegative integer coefficients and the leading coefficient of Q equals $a(G)$, the number of acyclic orientations of G.

A permutation $p(1)p(2)\cdots p(n)$ of the set $\{1, 2, \ldots, n\}$ is said to have an ascent at $p(i+1)$ if $p(i)<p(i+1)$. By definition any permutation $p \in S_n$ has an ascent at $p(1)$.

Proposition 1. *The polynomial* $Q(x)=\sum_{p=k}^{n} w_p x^p$, *where* $k=\chi(G)$ *and its coefficients* w_p *satisfy the following properties*:

(i) $w_n=a(G)$, $w_k=k!\,\mathrm{Col}_k(G)$ *and* $w_k+w_{k+1}+\cdots+w_n=n!$;

(ii) $$w_p=P(p)-\binom{n+1}{1}P(p-1)+\binom{n+1}{2}P(p-2)-\cdots+(-1)^{p-k}\binom{n+1}{p-k}P(k), \quad \textit{for } k \leqslant p \leqslant n;$$

(iii) $w_{n-r} = (-1)^n\left[P(-r-1) - \binom{n+1}{1}P(-r) + \binom{n+1}{2}P(-r+1) - \cdots + (-1)^r\binom{n+1}{r}P(-1)\right]$, for $0 \leqslant r \leqslant n-k$,

(iv) $P(x) = w_n\binom{x}{n} + w_{n-1}\binom{x+1}{n} + \cdots + w_k\binom{x+n-k}{n}$.

Proof. Since $P_G(x) = \sum_{p=k}^{n} (x)_p \operatorname{Col}_p(G)$ it follows that

$$F_G(x) = \sum_{n=0}^{\infty} P(n)x^n = \sum_{p=k}^{n}\sum_{n=0}^{\infty} (n)_p \operatorname{Col}_p(G)x^n = \sum_{p=k}^{n} \operatorname{Col}_p(G) \sum_{n=0}^{\infty} (n)_p x^n$$
$$= \sum_{p=k}^{n} \operatorname{Col}_p(G)x^p \frac{\mathrm{d}^p}{\mathrm{d}x^p}\left(\frac{x^p}{1-x}\right) = \sum_{p=k}^{n} \operatorname{Col}_p(G)x^p \frac{\mathrm{d}^p}{\mathrm{d}x^p}\left(\frac{1}{1-x}\right)$$
$$= \sum_{p=k}^{n} \frac{\operatorname{Col}_p(G)p!\,x^p}{(1-x)^{p+1}},$$

hence

$$Q(x) = \sum_{p=k}^{n} \operatorname{Col}_p(G)p!\,x^p(1-x)^{n-p} \tag{1}$$

(i) From (1) it follows that $w_k + \cdots + w_n = Q(1) = n!$, since $\operatorname{Col}_n(G) = 1$, $w_k = k!\operatorname{Col}_k(G)$ and $w_n = (-1)^n P(-1) = a(G)$ by [3].

(ii) We obtain from (1) that

$$w_p = \sum_{r=k}^{p} (-1)^{p-r} \operatorname{Col}_r(G)r!\binom{n-r}{p-r},$$

and by the inverse binomial formulas $\operatorname{Col}_r(G)r! = \sum_{s=k}^{r} (-1)^{r-s}\binom{r}{s}P(s)$. Hence

$$w_p = \sum_{s=k}^{p} (-1)^{p-s} \sum_{r=s}^{p} \binom{n-r}{p-r}\binom{r}{s}P(s) = \sum_{s=k}^{p} (-1)^{p-s}\binom{n+1}{p-s}P(s)$$

by standard binomial formulas.

(iv) follows from the fact that

$$\sum_{p=0}^{\infty} \binom{p+r}{n}x^p = \frac{x^{n-r}}{(1-x)^{n+1}}$$

for any $r \geqslant 0$, i.e., the both sides of this equality have the same generating function.

(iii) may be deduced from (iv) by induction on $r \geqslant 0$ as follows: For $r = 0$ we obtain $w_n = (-1)^n P(-1)$. Suppose that (iii) is valid for any $0 \leqslant r \leqslant p-2$. For $x = -p$ in (iv) we can write

$$P(-p) = w_n\binom{-p}{n} + w_{n-1}\binom{-p+1}{n} + \cdots + w_{n-(p-1)}\binom{-1}{n},$$

hence

$$w_{n-(p-1)}=(-1)^n\left[P(-p)-\sum_{s=0}^{p-2}\binom{p+n-(s+1)}{n}w_{n-s}\right]$$
$$=(-1)^n\left[P(-p)-\sum_{s=0}^{p-2}\binom{p+n-(s+1)}{n}\sum_{i=0}^{s}(-1)^i\binom{n+1}{i}P(-s+i-1)\right]$$

by the induction hypothesis. Hence

$$w_{n-(p-1)}=(-1)^n\left[P(-p)+\sum_{i=0}^{p-2}S_iP(-p+i+1)\right],$$

where

$$S_i=\sum_{j=0}^{i}(-1)^{j-1}\binom{n+1}{j}\binom{n+i-j+1}{n}.$$

We must show that $S_i=(-1)^{i+1}\binom{n+1}{i+1}$ for $i=0,\ldots,p-2$, or equivalently, that

$$\sum_{j=0}^{i+1}(-1)^j\binom{n+1}{j}\binom{n+i-j+1}{n}=0.$$

But this binomial identity follows by equating the coefficients of x^{i+1} in the both sides of the identity $(1-x)^{n+1}(1-x)^{-(n+1)}=1$ ([4], p. 299). $\square$

A combinatorial interpretation of the numbers $(-1)^nP(-r)$ is given in [3].

For the graph $\bar{K}_n$ composed from n isolated vertices one deduces that $P_{\bar{K}_n}(x)=x^n$ and $k=1$. From (ii) it follows that in this case $w_p=A(n,p)$, the Eulerian number with parameters n and p counting the number of permutations in S_n having p ascents and (iv) becomes Worpitzky's formula for Eulerian numbers.

A combinatorial interpretation of the numbers w_p for $p=0,\ldots,n$ may be stated as follows: Let G be a graph with vertex set $\{1,2,\ldots,n\}$ and $A(G)$ the set of all acyclic orientations of G, i.e., if $D\in A(G)$, then D is obtained by orienting all edges from $E(G)$ such that the digraph having vertex set $\{1,2,\ldots,n\}$ and arc set D contains no directed cycles. D may be regarded as a binary relation $>$ on $V(G)$ defined by $u>v$ if $(u,v)\in D$. Because D is acyclic, the transitive and reflexive closure $\bar{D}$ of D is a partial ordering of $V(G)$.

For a fixed acyclic orientation D perform the following construction: Consider first a bijective coloring $\varphi:V(G)\to\{1,2,\ldots,n\}$ compatible with D, i.e., $x>y$ implies $\varphi(x)>\varphi(y)$ and the set $\mathrm{TO}(\bar{D})$ of all total orders on $\{1,2,\ldots,n\}$ which extend $\bar{D}$.

For any total order $\mu=i_1<i_2<\cdots<i_n$ in $\mathrm{TO}(\bar{D})$ consider the permutation $\varphi(\mu)=\varphi(i_1)\varphi(i_2)\cdots\varphi(i_n)\in S_n$ and the set $\bigcup_{\mu\in\mathrm{TO}(\bar{D})}\varphi(\mu)$. It is clear that this set of permutations does not depend on φ, i.e., if ψ is another bijective coloring $\psi:V(G)\to\{1,2,\ldots,n\}$ compatible with D, then $\bigcup_{\mu\in\mathrm{TO}(\bar{D})}\varphi(\mu)=$

$\bigcup_{\mu \in TO(\bar{D})} \psi(\mu)$. The collection of all permutations in $\bigcup_{\mu \in TO(\bar{D})} \varphi(\mu)$ whenever $D \in A(G)$ is a multiset composed from $n!$ permutations in S_n, which will be denoted $M(G)$.

For example, for the cycle C_4 the set $A(D)$ contains 14 acyclic orientations: 8, 4 and 2 orientations containing a path of maximal length equal to 3, 2 and 1, respectively. In this case

$$M(G) = \{1234^{(14)}, 1324^{(4)}, 2134^{(2)}, 1243^{(2)}, 2143^{(2)}\},$$

where the exponent indicates the multiplicity. We obtain

$$P_{C_4}(x) = (x-1)^4 + x - 1 \quad \text{and} \quad F_{C_4}(x) = \frac{14x^4 + 8x^3 + 2x^2}{(1-x)^5}.$$

Also $M(K_n) = e^{(n!)}$, where e is the identity in S_n and $F_{K_n}(x) = n!x^n/(1-x)^{n+1}$.

Proposition 2. *For any graph G the coefficient w_p equals the number of permutations in $M(G)$ having p ascents for any $p \geq 0$.*

Proof. If P is a p-element partially ordered set, define the strict order polynomial $\bar{\Omega}(P, x)$ (evaluated at the nonnegative integer x) to be the number of strict order-preserving maps $\sigma: P \to \{1, 2, \ldots, x\}$, i.e., if $u < v$ in P, then $\sigma(u) < \sigma(v)$. From [2] it follows that

$$\sum_{n=0}^{\infty} \bar{\Omega}(P, n)x^n = \frac{\sum_{s=1}^{p} w_s^p x^s}{(1-x)^{p+1}},$$

where the coefficient w_s^p is equal to the number of permutations in $\bar{\mathscr{L}}(P, \varphi)$ with exactly s ascents. The φ-separator $\bar{\mathscr{L}}(P, \varphi)$ (in terminology of [2]) is defined as follows: φ is a bijective labeling $\varphi: P \to \{1, 2, \ldots, p\}$ compatible with P, i.e., $x < y$ in P implies $\varphi(x) < \varphi(y)$ and $\bar{\mathscr{L}}(P, \varphi) = \bigcup_{\mu \in TO(P)} \varphi(\mu)$, where $TO(P)$ is the set of total orders which extend P. Because $P_G(x) = \sum_{D \in A(G)} \bar{\Omega}(\bar{D}, x)$ ([3]), it follows that $w_p = \sum_{D \in A(G)} w_p^{\bar{D}}$, i.e., w_p is equal to the number of permutations in $M(G)$ having p ascents. $\square$

For $G = \bar{K}_n$ one deduces that $M(G) = S_n$, $w_p = A(n, p)$ for any p and $Q(x) = xE_n(x)$, where $E_n(x)$ is the classical Eulerian polynomial. Thus the numbers w_p occuring in the chromatic generating function of a graph G appear to be a graphical generalization of the Eulerian numbers.

References

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